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The effect of a uniform vertical magnetic field on the linear growth rates of steady Marangoni convection in a horizontal layer of conducting fluid

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Abstract

In this paper we investigate the effect of a uniform vertical magnetic field on the linear growth (and decay) rates of steady Marangoni convection in a horizontal layer of electrically-conducting fluid heated from below. Explicit analytical expressions for the linear growth rates of both long- and short-wave instability modes are derived for the first time. Numerically-calculated results for the linear growth rates are also presented. In particular, we show that the effect of increasing the magnetic field strength is always to stabilise the layer by decreasing the growth rates of the unstable modes. © 1998 Elsevier Science Ltd. All rights reserved.

Nomenclature

a total horizontal wave number
Bi Biot number
Bo Bond number
C_r crispation number
d initial thickness of the layer
f magnitude of free surface deflection
g gravitational acceleration
h heat transfer coefficient
$h_z(z)$ vertical variation of vertical magnetic field perturbation
H Hartmann number
$\bar{H}$ initial magnetic field strength
k thermal conductivity
M Marangoni number
P_1 Prandtl number
P_2 magnetic Prandtl number
s temporal growth rate
t time
$T(z)$ vertical variation of temperature perturbation
$w(z)$ vertical variation of vertical velocity perturbation
x, y, z spatial Cartesian coordinates.

Greek symbols

β initial temperature gradient
$-\gamma$ coefficient of thermal surface tension variation
η electrical resistivity
κ thermal diffusivity
μ magnetic permeability
ν kinematic viscosity of fluid
ρ density of fluid
σ electrical conductivity
τ_0 initial value of surface tension.

Subscript

c critical state.

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1. Introduction

The onset of thermocapillary-driven (Marangoni) convection in a layer of fluid heated from above or below is a fundamental model problem for several material processing technologies, most notably semiconductor crystal growth from a melt in microgravity conditions where, as Schwabe [1] describes, typically thermocapillary forces dominate buoyancy forces.

In the absence of a magnetic field, Pearson [2] showed that thermocapillary effects will drive steady Marangoni convection in a fluid layer of finite depth heated from below provided that the Marangoni number M exceeds a critical value M_c. Pearson's [2] work was restricted to the limit of strong surface tension in which the free upper surface is non-deformable. Scriven and Sternling [3], Smith [4] and Takashima [5] extended Pearson's [2] analysis to include the effect of free-surface deformation and showed that the onset of steady Marangoni convection can be as either a long-wave ($a_c = 0$) or a short-wave ($a_c = O(1)$) mode, where a_c is the critical wave number.

In many practical applications (such as crystal growth) the onset of convection is undesirable, and as a consequence there has been considerable interest in understanding various additional physical mechanisms for delaying, or possibly eliminating altogether, the onset of convection. In this paper we shall be concerned with one such mechanism, namely an externally-imposed uniform vertical magnetic field.

The effect of a uniform vertical magnetic field on the onset of Marangoni convection in a horizontal layer of electrically-conducting fluid heated from below was first addressed by Nield [6]. Nield [6] studied the onset of steady Marangoni convection in the case of a non-deformable free surface and showed that increasing the magnetic field strength has the stabilising effect of monotonically increasing the critical Marangoni number for the onset of convection. Subsequently Maekawa and Tanasawa [7] studied the same problem with an inclined magnetic field and concluded that only the vertical component of the magnetic field has any effect on the critical Marangoni number. In a series of papers Sarma [8–13] analysed the effect of both uniform rotation of the layer and a uniform vertical magnetic field on the onset of steady Marangoni convection in a horizontal fluid layer with a deformable free surface for a variety of combinations of thermal and magnetic boundary conditions. Unfortunately, he used an incorrect normal stress boundary condition at the free surface in his analysis, and so all his results for situations with a non-zero magnetic field and a deformable free surface are incorrect. Wilson [14–16] obtained the correct form of this boundary condition and used it to give a comprehensive description of the onset of steady convection in the presence of a magnetic field. Specifically, Wilson [14–16] showed that the effect of increasing the magnetic field strength is always a stabilising one but, in contrast to the results in the case of a non-deformable free surface, an arbitrarily large magnetic field cannot stabilise all disturbance when the free surface is deformable. The spatial structure of the convection in the limit of large magnetic field strength was subsequently examined by Nitschke and Thess [17].

All these works were concerned with the onset of steady convection in a layer heated from below.[1] In an important paper, Kaddame and Lebon [21] re-examined the simplest case of a layer with a non-deformable free surface and perfectly electrically conducting boundaries in which the onset of convection is always steady in the absence of the magnetic field. Kaddame and Lebon [21] demonstrated that in this case there are situations in which oscillatory convection not only occurs but is actually preferred to steady convection at the onset of instability. Unfortunately, Kaddame and Lebon's [21] solution is not entirely correct and so their results should be treated with some caution. However, this problem was recently re-examined by Hashim and Wilson [22] who determined the regions of $P_1 - P_2 - H$ parameter space where oscillatory convection is preferred to steady convection. In particular, Hashim and Wilson [22] confirmed Kaddame and Lebon's [21] conclusion that oscillatory Marangoni convection is only possible if $P_1 < P_2$, where P_1 and P_2 are the Prandtl number and magnetic Prandtl number, respectively. However, in most practical situations $P_1 \gg P_2$ and so the onset of convection will normally be steady as assumed by the earlier authors.

All of the works described above concentrated on the determination of the marginal stability curves for the onset of convection. There has been much less work on the temporal growth (and decay) rates of the instability. Regnier and Lebon [23] recently investigated the linear growth rates for both long- and short-wave modes for the pure Marangoni problem near the onset of convection. In particular, Regnier and Lebon [23] showed that free-surface deformation has a strong influence on the growth rates of the long-wave mode but has only a weak effect on the growth rates of the short-wave mode. Subsequently, Wilson and Thess [24] studied the linear growth rates of long-wave modes without the restriction of near critical conditions for the coupled Bénard–Marangoni problem including the effects of buoyancy in the bulk of the fluid. These analytical results were found to be in good agreement with the experimental observations of VanHook et al. [25]. There has been no work so far on the effect of a magnetic field on the linear growth rates of either the long- or short-wave modes.

In this paper we derive for the first time explicit analytical expressions for the linear growth (and decay) rates of both the long- and short-wave modes of Marangoni

[1] In contrast, when the layer is heated from above Takashima [18] showed that in the absence of a magnetic field convection only occurs when the free surface is deformable and that the preferred mode of convection is always oscillatory. Nitschke et al. [19] and Wilson [14, 20] studied the effect of a magnetic field on the onset of convection in this case.

convection in a horizontal layer of electrically-conducting fluid heated from below subject to a uniform vertical magnetic field. We also present numerically-calculated results for the linear growth rates. This work is a natural extension of previous studies which concentrated on the effect of a magnetic field on the marginal stability curves, and the recent work by Regnier and Lebon [23] and Wilson and Thess [24] on the linear growth rates in the absence of magnetic effects.

2. Linearised problem

The linearised equations and boundary conditions governing the onset of Marangoni convection in an initially quiescent horizontal layer of electrically-conducting fluid with a deformable free surface subject to a uniform vertical magnetic field and a uniform vertical temperature gradient are given by

$$(D^2-a^2-sP_1)T+w=0 \tag{1}$$

$$(D^2-a^2-sP_2)h_z+Dw=0 \tag{2}$$

$$(D^2-a^2)[(D^2-a^2-s)w+H^2Dh_z]=0 \tag{3}$$

subject to

$$sf-w=0 \tag{4}$$

$$P_1C_r[(D^2-3a^2-H^2-s)Dw + sP_2H^2h_z]-a^2(a^2+Bo)f=0 \tag{5}$$

$$P_1(D^2+a^2)w+a^2M(P_1T-f)=0 \tag{6}$$

$$h_z=0 \tag{7}$$

$$P_1DT+Bi(P_1T-f)=0 \tag{8}$$

evaluated on the undisturbed position of the upper free surface $z=1$, and

$$w=0 \tag{9}$$

$$Dw=0 \tag{10}$$

$$h_z=0 \tag{11}$$

evaluated on the lower rigid boundary $z=0$, together with either

$$T=0 \tag{12}$$

on $z=0$ if the boundary is conducting to temperature perturbations or

$$DT=0 \tag{13}$$

on $z=0$ if the boundary is insulating to temperature perturbations. The variables $w=w(z)$, $T=T(z)$, $h_z=h_z(z)$ and f denote the vertical variation of the z-component of the velocity, temperature and the z-component of the magnetic field, and the magnitude of the free surface deflection of the linear perturbation to the basic state with total wave number a in the horizontal x–y plane and complex growth rate s. The operator $D=\mathrm{d}/\mathrm{d}z$ denotes differentiation with respect to the vertical coordinate z. We have non-dimensionalised the variables using the scales d, d^2/ν, ν/d, $\beta d\nu/\kappa$, and $\mu\bar{H}/\eta$ for length, time, velocity, temperature, and magnetic field, respectively. The non-dimensional groups appearing in the problem are the Marangoni number $M=\gamma\beta d^2/\rho\nu\kappa$, the crispation number $C_r=\rho\nu\kappa/\tau_0 d$, the Hartmann number (the square root of the Chandrasekhar number) $H=\mu\bar{H}d(\sigma/\rho\nu)^{1/2}$, the Biot number $Bi=hd/k$, the Bond number $Bo=\rho g d^2/\tau_0$, the Prandtl number $P_1=\nu/\kappa$ and the magnetic Prandtl number $P_2=\nu/\eta$, where the symbols g, β, τ_0, $-\gamma$, ρ, ν, h, κ, k, μ, σ, η and $\bar{H}$ are defined in the Nomenclature section. Note that this choice of scaling was made for consistency with the work of Kaddame and Lebon [21] and Hashim and Wilson [22], but differs from that of Sarma [8–13] and Wilson [14–16, 20] who used the notation $P_r=P_1$ for the Prandtl number and $P_m=P_1/P_2$ for an alternative magnetic Prandtl number. However, the latter formulation can easily be recovered by multiplying the present time, temperature and magnetic field variables by $1/P_1$, P_1 and P_2, respectively.

Note that we can use equation (2) to rewrite equation (3) in the form

$$[(D^2-a^2)(D^2-a^2-s)-H^2D^2]w+sP_2H^2Dh_z=0. \tag{14}$$

Extensive use has been made of the symbolic algebra package MAPLE V (Release 3) running on a SUN SPARCstation 1+ to carry out much of the tedious algebraic manipulations needed in the course of finding analytical solutions. In this paper we shall mostly consider the case $P_2=0$, in which case the magnetic field h_z can be eliminated from the problem and hence the two boundary conditions on h_z [(7) and (11)] are therefore not required.

3. Growth rates of the long-wave ($a_c=0$) mode

In the absence of a useful closed form analytical solution to the full linear-stability problem we can obtain analytical expressions for the growth rates of the steady long wave ($a_c=0$) modes in the case $P_2=0$. Following Wilson and Thess [24] we seek asymptotic solutions for w, T and s in the forms

$$w=w_0(z)+a^2w_1(z)+O(a^4) \tag{15}$$

$$T=\frac{1}{a^2}T_0(z)+T_1(z)+O(a^2) \tag{16}$$

$$s=s_0+a^2s_1+a^4s_2+O(a^6) \tag{17}$$

in the long-wave limit $a\to 0$. Clearly if $s_0\neq 0$ then the leading order solutions of equations (1) and (14) take the form

$$w_0=A_1+A_2z+A_3\,e^{\xi_1 z}+A_4\,e^{-\xi_1} \tag{18}$$

$$T_0 = A_5\, e^{\xi_2 z} + A_6\, e^{-\xi_2 z} \tag{19}$$

while in the special case $s_0 = 0$ the solutions are simply

$$w_0 = A_1 + A_2 z + A_3\, e^{\xi_3 z} + A_4\, e^{-\xi_3 z} \tag{20}$$

$$T_0 = A_5 + A_6 z \tag{21}$$

where for convenience we have written $\xi_1^2 = s_0 + H^2$, $\xi_2^2 = s_0 P_1$ and $\xi_3 = H$ and A_i for $i = 1, \ldots, 6$ are arbitrary constants.

3.1. Conducting case

If $C_r = 0$ then the free surface is non-deformable. Substituting the expressions for w_0 and T_0 given in equations (18) and (19) into the leading order versions of the boundary conditions yields the appropriate solutions for $A_1, \ldots, A_6$ and the equation

$$(\xi_1 \cosh \xi_1 - \sinh \xi_1)(\xi_2 \cosh \xi_2 + Bi \sinh \xi_2) = 0 \tag{22}$$

and so either $s_0 = -(\xi^2 + H^2)$ where ξ is a root of $\tan \xi = \xi$ (in which case $T_0 \equiv 0$) or $s_0 = -\xi^2/P_1$ where ξ is a root of $\xi \cos \xi + Bi \sin \xi = 0$. The special case $s_0 = 0$ yields only the trivial solution and so in all cases we have $s_0 < 0$ and all the long-wave modes are stable as expected.

If $C_r \neq 0$ then the free surface is deformable. Using the expression for w_0 and T_0 given in equations (18) and (19) yields

$$\cosh \xi_1 (\xi_2 \cosh \xi_2 + Bi \sinh \xi_2) = 0 \tag{23}$$

and so either $s_0 = -(\xi^2 + H^2)$ where ξ is a root of $\cos \xi = 0$ (in which case again $T_0 \equiv 0$) or $s_0 = -\xi^2/P_1$ where ξ is a root of $\xi \cos \xi + Bi \sin \xi = 0$ as before. Once again all these modes have $s_0 < 0$ and so are stable.

In addition the special case $s_0 = 0$ now also yields a non-trivial solution. Using the expressions for w_0 and T_0 given in equations (20) and (21) we obtain

$$s_1 = \frac{MC_r H(C-1) - Bo(HC-S)(1+Bi)}{P_1 C_r H^3 C(1+Bi)} \tag{24}$$

where for convenience we have written $S = \sinh H$ and $C = \cosh H$. These long-wave modes will either grow or decay depending on whether $M > M_0$ or $M < M_0$, where M_0 is the value of the marginal stability curve at $a = 0$. Setting $s_1 = 0$ we recover the leading order behaviour of the steady marginal stability curve in the limit $a \to 0$ obtained by Wilson [14],

$$M_0 = \frac{Bo(S-HC)(1+Bi)}{C_r H(1-C)}. \tag{25}$$

In the limit $H \to 0$ we obtain

$$s_1 \sim \frac{3MC_r - 2Bo(1+Bi)}{6P_1 C_r (1+Bi)} \tag{26}$$

in agreement with the result of Wilson and Thess [24] for the non-magnetic case, while in the limit $H \to \infty$ we obtain

$$s_1 \sim \frac{MC_r - B_0(1+Bi)}{P_1 C_r (1+Bi)} \frac{1}{H^2}. \tag{27}$$

3.2. Insulating case

As in the conducting case there are stable long-wave modes and these are given by $s_0 = -(\xi^2 + H^2)$ where ξ is a root of $\tan \xi = \xi$ if $C_r = 0$ or $\cos \xi = 0$ if $C_r \neq 0$ (in both cases with $T_0 \equiv 0$) and $s_0 = -\xi^2/P_1$ where ξ is a root of $\xi \sin \xi - Bi \cos \xi = 0$.

In addition the special case $s_0 = 0$ yields a non-trivial solution when $Bi = 0$, and using the expressions for w_0 and T_0 given in equations (20) and (21) we obtain

$$\begin{aligned} &2H^2[(S-HC)Bo + \{M(C-1) - P_1 s_1 H^2 C\} HC_r] A_4 \\ &\quad + [(S+C-H-1)Bo - P_1 s_1 C_r H^3] MA_5 = 0. \end{aligned} \tag{28}$$

In order to determine s_1 it is necessary to proceed to next order in a^2 and using the appropriate expressions for w_1 and T_1 yields

$$\begin{aligned} &[\{4(C-HS-1) + H^2(C+1)\} Bo \\ &\quad + (H-S) 2P_1 s_1 H^3 C_r] A_4 \\ &\quad - [(S+C-H-1)Bo - P_1 s_1 C_r H^3] \\ &\quad \times (1 + P_1 s_1) HA_5 = 0 \end{aligned} \tag{29}$$

and so s_1 satisfies the quadratic equation

$$\begin{aligned} &4H^6 C_r C P_1^2 s_1^2 \\ &\quad - H^3[-4(S-HC)C_r M - 4H^3 C_r C \\ &\quad + 4(S-HC)Bo] P_1 s_1 - [(8C - 8HS \\ &\quad + 2H^2 C - 8 + 2H^2) Bo \\ &\quad + 4C_r H^4 (C-1)] M - 4H^3\, Bo(S-HC) = 0 \end{aligned} \tag{30}$$

which holds for all values of C_r and Bo. In the special case $C_r = 0$ equation (30) reduces to

$$s_1 = \frac{[(C+1)H^2 - 4HS + 4C - 4]M + 2H^3(S-HC)}{2H^3 P_1 (HC-S)}. \tag{31}$$

Setting $s_1 = 0$ we recover the leading behaviour of the marginal stability curve obtained by Wilson [16],

$$M_0 = \frac{2H^3(HC-S)Bo}{[H^2(1+C) - 4(HS-C+1)]Bo + 2H^4(C-1)C_r} \tag{32}$$

while in the limit $H \to \infty$ we obtain $s_1 \sim -1/P_1$.

The special case $s_0 = 0$ also yields a non-trivial solution when $Bi \neq 0$ if $C_r \neq 0$. If $Bo = 0$ then solving for w_0 and T_0 gives $w_0 \equiv 0$ and $s_1 = 0$. In order to determine s_2 it is again necessary to proceed to next order in a^2, and using the appropriate solutions for w_1 and T_1 we obtain

$$s_2 = \frac{MC_r H(C-1) + Bi(S-HC)}{H^3 P_1 C_r\, Bi\, C}. \tag{33}$$

Setting $s_2 = 0$ we recover the leading order behaviour of the steady marginal curve in the limit $a \to 0$ obtained by Wilson [16],

$$M_0 = \frac{Bi(HC-S)}{C_r H(C-1)} \tag{34}$$

and in the limit $H \to \infty$ we obtain

$$s_2 \sim \frac{MC_r - Bi}{P_1 C_r Bi} \frac{1}{H^2}. \tag{35}$$

If $Bo \neq 0$ then using the expressions for w_0 and T_0 given in equations (20) and (21) we obtain

$$s_1 = \frac{Bo(S-HC)}{H^3 P_1 C_r C}. \tag{36}$$

In the limit $H \to \infty$ we obtain

$$s_1 \sim -\frac{Bo}{P_1 C_r} \frac{1}{H^2}. \tag{37}$$

Note that in this case $s_1 < 0$ for all values of $H \geqslant 0$ and so all long-wave modes are always stable as expected.

In all cases we recover the corresponding expressions obtained by Wilson and Thess [24] for the non-magnetic problem in the limit $H \to 0$.

4. Growth rates of the short-wave ($a_c = O(1)$) modes

In this section we analyse the effect of the magnetic field on the linear growth rate of the steady short-wave modes. Regnier and Lebon [23] showed that in the absence of magnetic effects free-surface deformation has only a weak effect on the growth rates of these modes, and so we choose to study the simplest case of a non-deformable free surface ($C_r = 0$). Again as in Section 3 we set $P_2 = 0$ and consider only the case when the lower boundary is conducting to temperature perturbations ($T = 0$ on $z = 0$). Recall that in this case the work of Kaddame and Lebon [21] and Hashim and Wilson [22] shows that the onset of convection will always be steady.

In this case the governing equations and boundary conditions become

$$[(D^2-a^2)(D^2-a^2-s)-H^2D^2]w = 0 \tag{38}$$

$$(D^2-a^2-sP_1)T+w = 0 \tag{39}$$

subject to

$$w = 0 \tag{40}$$

$$(D^2+a^2)w+a^2MT = 0 \tag{41}$$

$$DT+Bi\,T = 0 \tag{42}$$

on $z = 1$, and

$$w = 0 \tag{43}$$

$$Dw = 0 \tag{44}$$

$$T = 0 \tag{45}$$

on $z = 0$.

In order to investigate the linear growth rates s near the instability threshold $s = 0$ we follow the approach of Regnier and Lebon [23] by seeking asymptotic solutions for w, T and s in the forms

$$w = w_0 + w_1\varepsilon + O(\varepsilon^2) \tag{46}$$

$$T = T_0 + T_1\varepsilon + O(\varepsilon^2) \tag{47}$$

$$s = s_1\varepsilon + O(\varepsilon^2) \tag{48}$$

in the limit $\varepsilon \to 0$, where the small parameter $\varepsilon = (M - M_c)/M_c$ measures the difference between the Marangoni number, M, and the critical value of the Marangoni number for the onset of steady convection, M_c.

4.1. Leading-order problem

At leading-order in ε the governing equations and boundary conditions are

$$[(D^2-a^2)^2-H^2D^2]w_0 = 0 \tag{49}$$

$$(D^2-a^2)T_0+w_0 = 0 \tag{50}$$

subject to

$$w_0 = 0 \tag{51}$$

$$(D^2+a^2)w_0+a^2M_cT_0 = 0 \tag{52}$$

$$DT_0+Bi\,T_0 = 0 \tag{53}$$

on $z = 1$ and

$$w_0 = 0 \tag{54}$$

$$Dw_0 = 0 \tag{55}$$

$$T_0 = 0 \tag{56}$$

on $z = 0$. The leading order problem may be written as

$$\mathbf{L}\mathbf{W}_0 = \mathbf{0} \tag{57}$$

where

$$\mathbf{L} = \begin{bmatrix} (D^2-a^2)^2-H^2D^2 & 0 & 0 \\ 1 & (D^2-a^2) & 0 \\ D^2|_{z=1} & 0 & a^2M_c \end{bmatrix},$$

$$\mathbf{W}_0 = \begin{bmatrix} w_0 \\ T_0 \\ T_0|_{z=1} \end{bmatrix} \tag{58}$$

subject to the boundary conditions (51), (53)–(56). Note that the boundary condition involving M_c has been included in the operator $\mathbf{L}$. Solving the system (57) subject to the remaining boundary conditions (51), (53)–(56) yields w_0, T_0 and the steady marginal stability curve studied by earlier authors, M_c. Since these expressions are rather lengthy they are not given here for brevity. Clearly we have to go to the next order in ε to determine s_1.

4.2. First-order problem

At first-order in ε the governing equations and boundary conditions are

$$(D^2-a^2)^2w_1-H^2D^2w_1-s_1(D^2-a^2)w_0=0 \quad (59)$$

$$(D^2-a^2)T_1+w_1-s_1P_1T_0=0 \quad (60)$$

subject to

$$w_1=0 \quad (61)$$

$$(D^2+a^2)w_1+a^2M_cT_1+a^2M_cT_0=0 \quad (62)$$

$$DT_1+Bi\,T_1=0 \quad (63)$$

on $z=1$ and

$$w_1=0 \quad (64)$$

$$Dw_1=0 \quad (65)$$

$$T_1=0 \quad (66)$$

on $z=0$. The first-order problem may be written as

$$\mathbf{L}\mathbf{W}_1=\mathbf{F} \quad (67)$$

where

$$\mathbf{W}_1=\begin{bmatrix} w_1 \\ T_1 \\ T_1|_{z=1}\end{bmatrix},\quad \mathbf{F}=\begin{bmatrix} s_1(D^2-a^2)w_0 \\ s_1P_1T_0 \\ -a^2M_cT_0|_{z=1}\end{bmatrix} \quad (68)$$

and the operator $\mathbf{L}$ is given in (58), subject to the boundary conditions (61), (63)–(66).

4.3. First-order adjoint problem

In principle, we can solve the first-order problem for s_1 directly, but since the algebraic manipulations involved are too complicated even for MAPLE V, we instead obtain s_1 indirectly by solving the first-order adjoint problem. Non-trivial solutions of the system (67), subject to the boundary conditions (61), (63)–(66), exist if and only if the Fredholm Alternative (see, for example, Friedman [26]) holds, i.e. if $\mathbf{F}$ is orthogonal to any solution of the homogeneous first-order adjoint problem

$$\mathbf{L}^*\mathbf{W}_1^*=\mathbf{0} \quad (69)$$

where the adjoint operator $\mathbf{L}^*$ is defined by

$$\langle \mathbf{W}_1^*, \mathbf{L}\mathbf{W}_1\rangle=\langle \mathbf{W}_1, \mathbf{L}^*\mathbf{W}_1^*\rangle \quad (70)$$

and $\langle\cdot,\cdot\rangle$ denotes the scalar product

$$\langle \mathbf{a},\mathbf{b}\rangle=\int_0^1(a_1b_1+a_2b_2)\,\mathrm{d}z+a_3b_3|_{z=1}. \quad (71)$$

Integration by parts of the left-hand side of equation (70) and use of the boundary conditions (61), (63)–(66) shows that if we take the adjoint solution to be

$$\mathbf{W}_1^*=\begin{bmatrix} w_1^* \\ T_1^* \\ Dw_1^*|_{z=1}\end{bmatrix} \quad (72)$$

then $\mathbf{L}^*$ is given by

$$\mathbf{L}^*=\begin{bmatrix} (D^2-a^2)^2-H^2D^2 & 1 & 0 \\ 0 & (D^2-a^2) & 0 \\ 0 & -(D+Bi)|_{z=1} & a^2M_c\end{bmatrix} \quad (73)$$

with the adjoint boundary equations

$$w_1^*=0 \quad (74)$$

$$D^2w_1^*=0 \quad (75)$$

on $z=1$ and

$$w_1^*=0 \quad (76)$$

$$Dw_1^*=0 \quad (77)$$

$$T_1^*=0 \quad (78)$$

on $z=0$. The orthogonality condition is expressed as

$$\langle \mathbf{F}, \mathbf{W}_1^*\rangle=0 \quad (79)$$

from which we obtain

$$\tau_0=s_1^{-1}=\frac{\displaystyle\int_0^1\{[(D^2-a^2)w_0]w_1^*+P_1T_0T_1^*\}\,\mathrm{d}z}{a^2M_cT_0|_{z=1}Dw_1^*|_{z=1}} \quad (80)$$

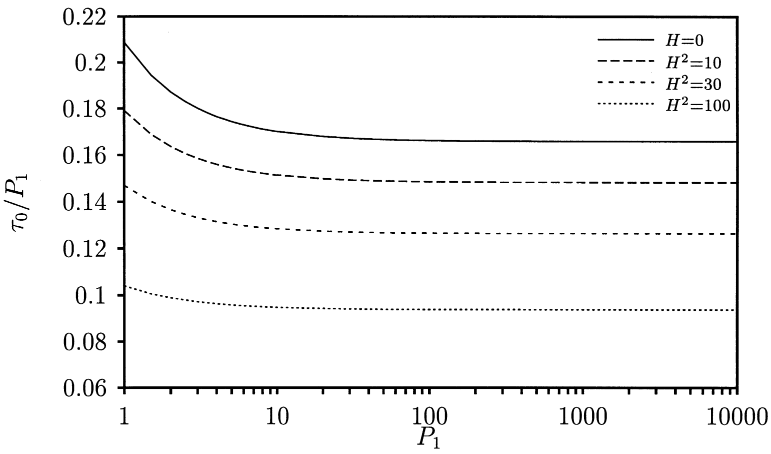
where τ_0 is called the relaxation time. In what follows we plot τ_0/P_1 (i.e. the relaxation time measured in units of d^2/κ) rather than τ_0 (i.e. the relaxation time measured in units of d^2/ν) for consistency with the earlier work of Regnier and Lebon [23]. In principle we can evaluate τ_0 analytically but the resulting expression is extremely lengthy and so is not repeated here. We can, however, evaluate τ_0 numerically. Note that in the special case $H=0$ we recover the corresponding results of Regnier and Lebon [23] for the non-magnetic problem.

Figures 1(a) and (b) show the numerically-calculated values of τ_0/P_1 plotted as a function of P_1 in the case $Bi=0$ for a range of values of H. As shown in Figs 1(a) and (b), the effect of increasing P_1 is to increase the relaxation time τ_0. Increasing H for a fixed value of P_1 has the effect of decreasing τ_0.

Figure 2 shows the numerically-calculated values of τ_0/P_1 plotted as a function of H in the case $Bi=0$ for a range of values of P_1. For a fixed value of P_1, increasing H has the effect of decreasing τ_0, and for a fixed value of H, increasing P_1 has the effect of increasing τ_0.

Figure 3 shows the numerically-calculated values of τ_0/P_1 plotted as a function of Bi in the case $P_1=10$ for a range of values of H. As Fig. 3 shows for a fixed value of H, τ_0 is a decreasing function of Bi. For a fixed value of Bi, τ_0 is a decreasing function of H.

(a)



(b)

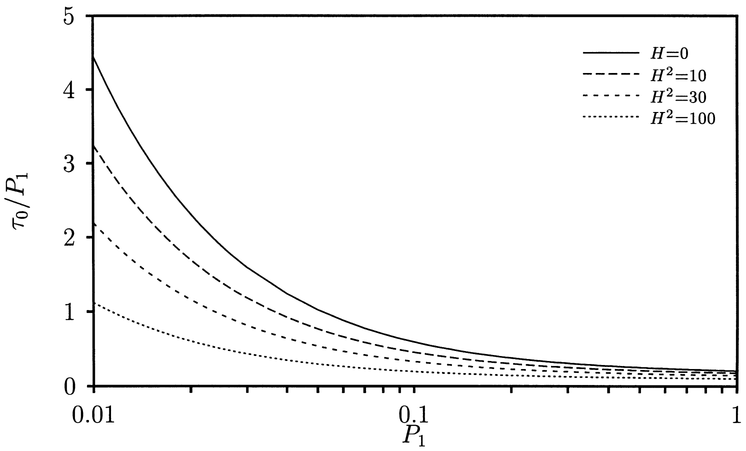


Fig. 1. Numerically-calculated relaxation time τ_0/P_1 of the short-wave mode as a function of P_1 in the case $C_r = 0$ and $Bi = 0$ for a range of values of H: (a) large P_1; and (b) small P_1.

5. Numerically-calculated growth rates

In general, in order to calculate the linear growth rates we have to turn to numerical computation to evaluate the complex determinant of the coefficients of the eight unknowns in the eight linear equations obtained by substituting the general solution of the governing equations (1)–(3) into the boundary conditions (4)–(13). This task was performed using NAG routine F03ADF incorporated into a FORTRAN program running on SUN SPARCstation 1+.

Figure 4 shows numerically-calculated values of Re(s), the real part of s, in the conducting case plotted as functions of a for a range of values of H in the case $M = 500$, $C_r = 0$, $Bi = 0$, $P_1 = 100$ and $P_2 = 1$. Note that in all the cases shown s is purely real. In particular, the results in Fig. 4 show that increasing H stabilises the layer both by decreasing the range of unstable modes for a given value of the Marangoni number and by decreasing the growth rates of the unstable modes.

6. Conclusions

In this paper we derived for the first time explicit analytical expressions for the linear growth (and decay) rates of both the long- and short-wave modes of Marangoni convection in a horizontal layer of electrically-conducting

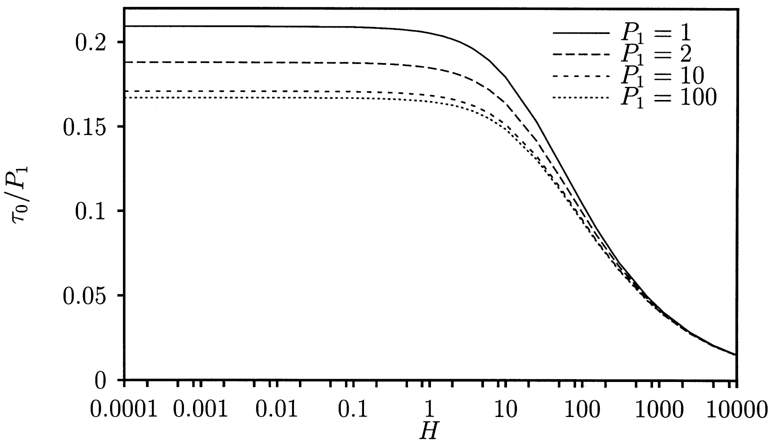


Fig. 2. Numerically-calculated relaxation time τ_0/P_1 of the short-wave mode as a function of H in the case $C_r = 0$, and $Bi = 0$ for a range of values of P_1.

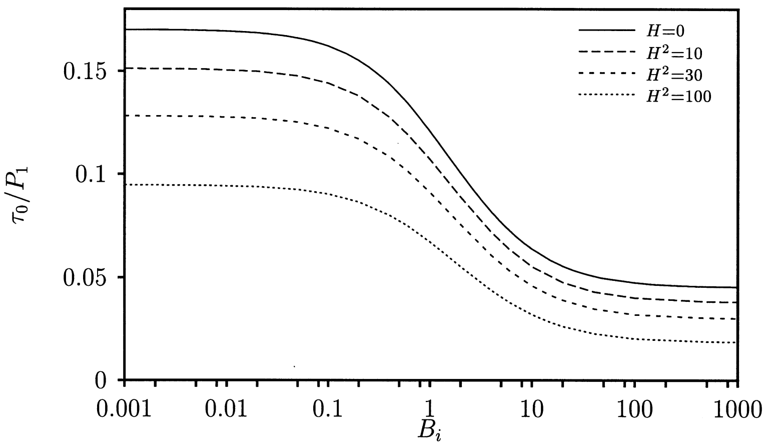


Fig. 3. Numerically-calculated relaxation time τ_0/P_1 of the short-wave mode as a function of Bi in the case $C_r = 0$ and $P_1 = 10$ for a range of values of H.

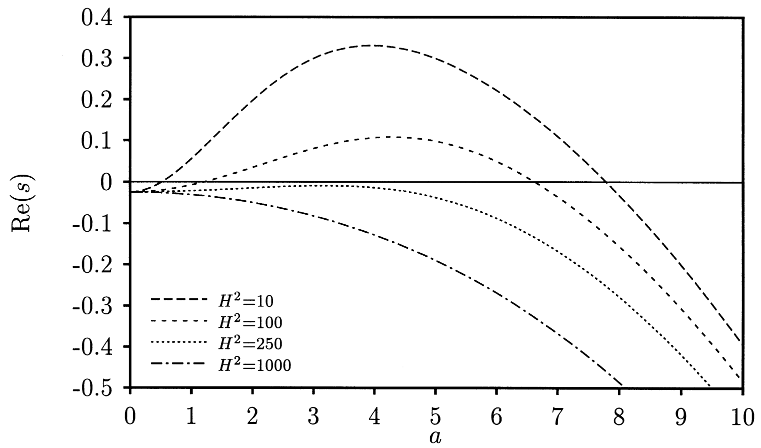


Fig. 4. Numerically-calculated values of Re(s) in the conducting case plotted as functions of a for a range of values of H in the case $M = 500$, $C_r = 0$, $Bi = 0$, $P_1 = 100$ and $P_2 = 1$. In all the cases shown the imaginary part of s is identically zero.

fluid heated from below subject to a uniform vertical magnetic field. We also presented numerically-calculated results for the linear growth rates. In particular, we showed that the effect of increasing H is always to stabilise the layer by decreasing the growth rates of the unstable modes.

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